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A mathematical approach to a forest-impact problem

ANATOLI LEONTIEV[1], RONALDO S. BUSSE[1] and WILMA HUACASI[2]
[1]*Instituto de Matemática, Universidade Federal do Rio de Janeiro, 21945 970, Rio de Janeiro-RJ, Brasil*
[2]*Departamento de Matemáticas-CCT, Universidade Estadual do Norte Fluminense, 28015 620, Campos dos Goytacazes-RJ, Brasil*

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Abstract. In this article a mathematical model of forests impact on aquifers is proposed. This phenomenon is the lowering of the groundwater table under areas covered by trees. The model includes a boundary-value problem with contact and free-boundary conditions. A variational formulation of this problem, which is a quasi-variational inequality, is obtained. Its equivalence with the original problem is proved; existence and uniqueness results are obtained. A numerical example of the model is given.

Key words: boundary-element method, porous media, quasivariational inequality, suction flux, unconfined steady flow.

1. Introduction

Various studies on groundwater flow realized during the last two centuries show increasing interest in the problem by virtue of the importance of water-resources management for the future of humanity. In the present paper we deal with the phenomenon of forest impact on aquifers. This problem appears in different forms in various fields of activity, such as agriculture, civil engineering, etc.

By forest impact on aquifers we mean the effect of lowering the groundwater table under areas covered by trees; see Figure 1. To study this phenomenon, the use of experimental methods is common; see [1–3], for example. The experiments consist in real-time and real-scaled monitoring of the water-table response under a forest area and can take many years. To predict the groundwater-level reduction, water-balance models are applied, as is done in [1].

From a hydromechanical point of view this phenomenon can be considered as a problem of unconfined flow in porous media with possible fluid discharge (evaporation) through the water table owing to suction of the tree roots. The location of the water table under the forest-suction effect, the flow characteristics, as well as the region of the contact of the aquifer with the tree-root system are the unknowns of this problem; see [3].

Mathematically, unconfined steady flow through porous media belongs to the category of free-boundary-value problems; see for example [4] or [5]. The problem is defined over the domain and a part of the contour, called free boundary which is unknown *a priori* and can be found as a component of the solution. We describe one problem of this class, especially the seepage (or dam) problem and some related results in Section 2.

Among the methods used to solve the unconfined steady-flow problems one may distinguish analytical, iterative and transformation methods. The analytical solution is obtained in [4] with the theory of analytical functions for linear ordinary differential equations. In the iterative method proposed in [6], one of two conditions defined at the free boundary is chosen

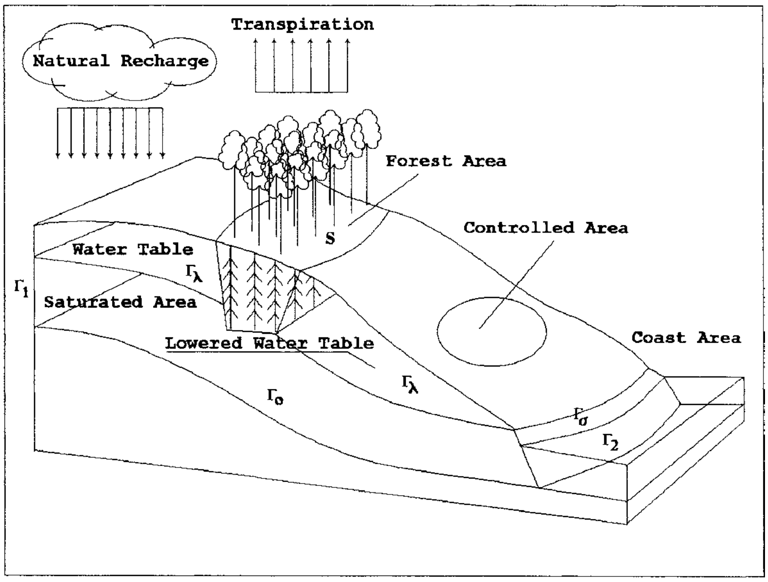

Figure 1. Water table and tree-roots system interaction scheme.

to solve, at each iteration, the direct value problem. By guessing an initial approximation, one adjusts the location of the free boundary at each iteration to make the other boundary condition hold; then the direct problem is re-solved, etc. The method of transformation is described in [7] and is known as the 'Baiocchi transformation". It consists in changing the problem variable to transform the free-boundary domain into a fixed domain. The problem on this new domain takes on the appearance of a variational inequality. The seepage problem can be considered also as a 'codimentional-two free-boundary problem' (see [8]) in which the only geometrical unknowns are the 'free points' which mark the points at which the free boundary meets the top of the dam. All these methods have been proposed to solve classical seepage problems, *i.e.*, problems without any evaporation (or infiltration) effect on the water table. Some problems with an *a piori* prescribed evaporation zone are considered in [9] and [10].

The two-dimensional model of the forest-impact phenomenon proposed in this paper includes a boundary-value problem with contact conditions that substitute for a part of the free-boundary conditions and is described in Section 3. In Section 4 we propose a Baiocchi-like transformation of the problem variables and obtain a quasivariational inequality associated with the boundary-value problem. In Section 5 we prove that this inequality is equivalent to the original boundary-value problem.

In Section 6 we construct a family of variational inequalities and indicate a sequence of its solutions that converge to the solution of our quasivariational inequality. We also prove that the solution of this quasivariational inequality is unique. Some methods for obtaining the existence and uniqueness results for different quasivariational inequalities can be found in [7], [11], 12]. As for the existence and uniqueness of the solution for variational formulations of various classical seepage problems, we refer the reader to [13].

An example involving a numerical implementation of our model is given in Section 7. The numerical technique is based on the shape-optimization approach. We transform the free-contact boundary problem into a least-squares-like shape-optimization problem. The objective functional contains one of the free-boundary conditions, whereas the state equation, together

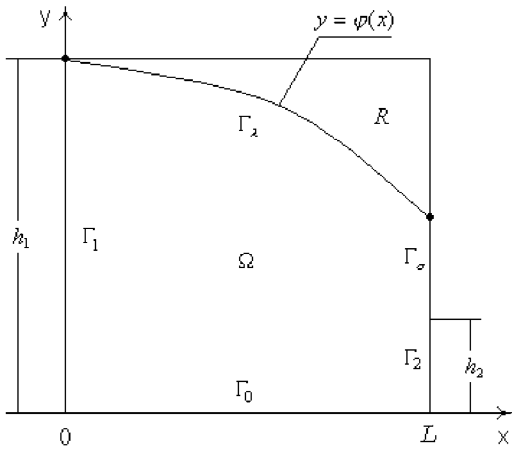

Figure 2. Unconfined fluid flow. Classical case.

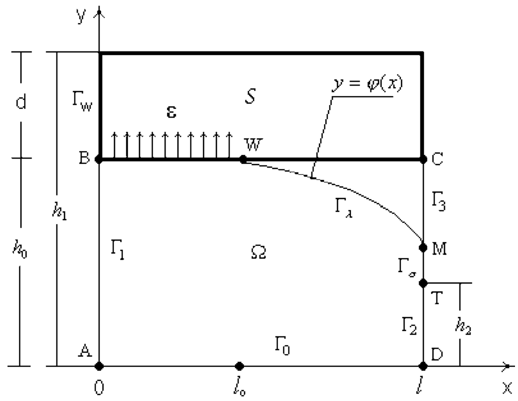

Figure 3. Unconfined fluid flow with suction.

with the other boundary conditions, become the problem constraints. We seek the minimum of the objective functional with respect to the shape of the water table. Performing a boundary-element discretization, we get a nonlinear mathematical programming problem. To solve it, we use Herskovits's interior-point algorithm described in [14]. The details of the numerical algorithm and boundary-element discretization for solving the forest-impact problem are presented in [15].

2. The classical seepage problem

In the classical case of unconfined flow through a porous medium, the unknowns of the problem are the characteristics of the flow, such as the velocity potential $u(x, y)$, and the flow region (aquifer) Ω itself; see [4, Chapter VII]. . A part of the aquifer boundary Γ_λ, called the water table, is unknown *a priori* and has to be located; see Figure 2.

In this paper we consider two-dimensional steady flow through a homogeneous and isotropic porous medium with the permeability coefficient $k = 1$ and assume that the external pressure is equal to zero. Let R be an open and, for the sake of convenience, rectangular domain occupied by the porous medium, h_1 and h_2 the fluid piezometric levels on the left and on the right sides of R, respectively, $\Gamma_\circ$ the impermeable bottom and Γ_σ the seepage line. The classical case does not assume any evaporation (or infiltration) effects on the water table.

Then, the classical seepage (dam) problem can be formulated as a free-boundary problem:

Problem 2.1. Find a potential $u(x, y)$ and a decreasing function $\varphi(x)$ that defines the location of the water table Γ_λ, satisfying

$$\Delta u = 0 \text{ in } \Omega, \quad u = h_1 \text{ on } \Gamma_1, \quad u = h_2 \text{ on } \Gamma_2,$$

$$u = y \text{ on } \Gamma_\sigma \cup \Gamma_\lambda, \quad q = 0 \text{ on } \Gamma_\circ \cup \Gamma_\lambda,$$

where $q \equiv \partial u/\partial n$ and n is the outward normal to $\Gamma_\circ \cup \Gamma_\lambda$.

In the unknown part Γ_λ of the boundary the function $u(x, y)$ has to fulfill two boundary conditions (free-boundary conditions) $u = y$ and $q = 0$. Thus, the water table is considered as a free boundary. Problem 2.1 admits an unique solution pair $\{\varphi, u\}$, where $\varphi(x)$ is smooth and $u \in H^1(\Omega) \cap C^\circ(\overline{\Omega})$; see [16, p. 237].

Performing the Baiocchi transformation $w(x, y) = \int_y^{\varphi(x)} (u(x,t) - t)\mathrm{d}t$, introduced in [7, Section 8.2], we obtain a variational inequality equivalent to Problem 2.1:

$$w \in K, \quad \int_R (w_x(v-w)_x + w_y(v-w)_y)\mathrm{d}x\mathrm{d}y \geq -\int_R (v-w)\mathrm{d}x\mathrm{d}y, \quad \forall v \in K.$$

Here $K = \{v \in H^1(R) \mid v \geq 0 \text{ in } R \text{ and } v = g \text{ on } \partial R\}$, where function g is defined by using values of h_1, h_2 and L, and the subscript x (or y) denotes the derivative with respect to x (or y). From the solution w of this inequality, the velocity potential is defined as $u = y - w_y$ and Γ_λ is determined as the curve that separates the areas with $w = 0$ and $w > 0$.

To obtain this variational formulation it is necessary to know the discharge across any vertical section of aquifer, $Q(x) \equiv -\int_0^{\varphi(x)} u_x(x,t)\mathrm{d}t$. In the classical case the Dupuit formula $Q(x) = (h_1^2 - h_2^2)/2L$ is used.

3. The forest-impact problem

The difference between the forest-impact problem and the classical seepage problem could be explained by the flow flux through the water table due to the aquifer reaching the tree-root system. Let $R \equiv ABCD$ be an open and, for convenience, rectangular domain occupied by the porous medium and S the tree-root system of depth $d > 0$; see Figure 3. The fluid is assumed to be ideal. The porous medium is homogeneous and isotropic with permeability coefficient equal to 1. The capillary effect on the water table is not considered in our model. We assume that suction flux with given rate $\varepsilon(x)$ is present in the part BW of the water table $\Gamma_\lambda \equiv BWM$ which reaches the bottom $S_\circ \equiv BC$ of the tree-root system. The left wall Γ_w of S is assumed impermeable. The contact area between aquifer and tree-root system BW is *a priori* unknown and can be defined together with the location of the rest of the water table *WM*, seepage *MT* and the velocity potential u in $\Omega \equiv ABWMTD$. We suppose also that the function $\varphi(x)$ defining the portion $\Gamma_\lambda \setminus S_\circ \equiv WM$ of the water table is decreasing.

For the forest-impact problem we define at the parts Γ_1, Γ_2, $\Gamma_\circ$ and Γ_σ of the boundary $\partial\Omega$ the same conditions as for Problem 2.1. The part of Γ_λ that does not contact S remains the free boundary and we apply here the conditions $u = y$ and $q = 0$. When $\Gamma_\lambda \cap S_\circ \neq \emptyset$ we have a flow with prescribed rate $\varepsilon(x)$ through this part of Γ_λ toward the interior of S. Then, we have the following:

Problem 3.1. Find potential $u(x, y)$ and decreasing function $\varphi(x)$ that define the portion of the water table Γ_λ without contact with S, satisfying

$$\Delta u = 0 \text{ in } \Omega, \quad u = h_1 \text{ on } \Gamma_1, \quad u = h_2 \text{ on } \Gamma_2,$$

$$u = y \text{ on } \Gamma_\sigma \cup (\Gamma_\lambda \setminus S_\circ), \quad q = 0 \text{ on } \Gamma_\circ \cup (\Gamma_\lambda \setminus S_\circ), \quad q = -\varepsilon(x) \text{ on } \Gamma_\lambda \cap S_\circ,$$

where $q \equiv \partial u/\partial n$ and n is the outward normal to $\Gamma_\circ \cup \Gamma_\lambda$.

At the water table we have conditions that take the form of free- or contact-boundary conditions. We call these ‘free-contact’ boundary conditions.

We assume that the discharge $Q(x)$ across any vertical section of aquifer (vertical discharge) for Problem 3.1 is such that

$$0 \le Q(x) \le \frac{h_1^2 - h_2^2}{2L}, \ \forall x \in [0, L] \tag{3.1}$$

and

$$Q'(x) = \begin{cases} -\varepsilon(x), & x \in [0, l^\circ], \\ 0, & x \in (l^\circ, L], \end{cases} \tag{3.2}$$

where $[0, l^\circ] \subset [0, L]$ is the interval that corresponds to the contact part of Γ_λ.

In addition to the model considered here, there are other situations where the flow through the water table is different from zero. This is the case for fluid flow with evaporation (or infiltration) through the water table. A variational formulation of the problem of unconfined fluid flow with an *a priori* given constant rate of evaporation through the whole water table is studied in [9], [10].

4. A quasivariational inequality associated to Problem 3.1

We assume that Problem 3.1 has a solution pair $\{\varphi, u\}$ with smooth function $\varphi(x)$ and $u \in H^1(\Omega) \cap C^\circ(\overline{\Omega})$ and introduce the following transformation:

$$w(x, y) = \int_y^{\psi(x)} (u(x, t) - t)\mathrm{d}t + w_\circ(x) \text{ in } \Omega, \tag{4.1}$$

where the function $\psi(x)$ represents the whole water table Γ_λ and the function $w_\circ(x)$ is defined in the following form:

$$\begin{aligned} &w_\circ \in C^1[0, L], \quad w_\circ(0) = d^2/2, \quad w_\circ(L) = 0, \\ &w_\circ''(x) = -\varepsilon(x), \ x \in [0, l^\circ] \ \text{ and } \ w_\circ''(x) = 0, \ x \in (l^\circ, L]. \end{aligned} \tag{4.2}$$

Using the boundary conditions of Problem 3.1 and formula (4.1), we obtain for the values of w on Γ_1, Γ_2 and Γ_σ, respectively:

$$w(0, y) = \int_y^{h_\circ} (h_1 - t)\mathrm{d}t + w_\circ(0) = \frac{(h_1 - y)^2}{2}, \quad y \in [0, h_\circ],$$

$$w(L, y) = \int_y^{h_2} (h_2 - t)\mathrm{d}t + w_\circ(L) = \frac{(h_2 - y)^2}{2}, \quad y \in [0, h_2],$$

$$w(L, y) = \int_y^{\psi(l)} (u(L, t) - t)\mathrm{d}t + w_\circ(L) = 0, \quad y \in [h_2, \varphi(L)].$$

Moreover,

$$w_x(x, 0) = \int_0^{\psi(x)} u_x(x, t)\mathrm{d}t + [u(x, \psi(x)) - \psi(x)]\psi'(x) + w_\circ'(x) = -Q(x) + w_\circ'(x).$$

Due to property (3.2) and definition (4.2), we have the following relation between $w_\circ'(x)$ and the vertical discharge $Q(x)$ corresponding to the velocity potential u:

$$-Q(x) + w_\circ'(x) = \text{Const} = -Q(0) + w_\circ'(0) \equiv -\frac{h_1^2 - h_2^2}{2L}. \tag{4.3}$$

Thus, on $\Gamma_\circ$ the function w is linear and

$$w(x,0) = \frac{h_1^2}{2} - \frac{h_1^2 - h_2^2}{2L} x.$$

Let us denote

$$w^\circ(x, y) \equiv w_\circ(x) \quad \text{for} \quad (x, y) \in R. \tag{4.4}$$

From (4.1) and using the smoothness of $\varphi(x)$, we have that $w_x = w'_\circ$ on $\Gamma_\lambda \cap R$. Thus, due to the boundary conditions of Problem 3.1, we can define an extension of class $C^1(R)$ of the function w, still called w:

$$w(x, y) = \begin{cases} w(x, y), & (x, y) \in \Omega, \\ w^\circ(x, y), & (x, y) \in R \setminus \Omega, \end{cases}$$

such that $|\text{grad} w| = |w'_\circ|$ in $R \setminus \Omega$.

The two following Lemmas are similar to the ones proved in [16, p. 229, p. 232].

Lemma 4.1. The function u satisfies $u(x, y) \geq y$ in Ω.

To prove this result we observe that $\int_{\Gamma_\lambda} \frac{\partial u}{\partial n} \xi \, \mathrm{d}\Gamma = 0$. Let us note that, physically, the inequality $u(x, y) \geq y$ in Ω means that the pressure of the fluid is nonnegative. From Lemma 4.1 and formula (4.1) it follows that $w \geq w^\circ$ in R.

Lemma 4.2. Let w is given by (4.1). Then, $\forall \xi \in C_\circ^\infty(R)$,

$$\int_R (w_x \xi_x + w_y \xi_y) \mathrm{d}x\mathrm{d}y + \int_R I_\Omega \xi \, \mathrm{d}x\mathrm{d}y = \int_{R\setminus\Omega} w_x \xi_x \, \mathrm{d}x\mathrm{d}y.$$

Let $G(x, y)$ be a function of class $C^1(\overline{R})$ such that $G = w$ on ∂R and $K_\circ$ a nonempty, convex and closed subset of $H^1(R)$:

$$K_\circ = \{v \in H^1(R) \mid v \geq w^\circ \text{ in } R \text{ and } v = G \text{ on } \partial R\}. \tag{4.5}$$

Since $w = w^\circ$, $v \geq w^\circ$ and $w_y = 0$ in $R \setminus \Omega$, using the Green formula and Lemma 4.2, we get:

$$\begin{aligned}
&\int_R (w_x (v - w)_x + w_y (v - w)_y) \mathrm{d}x\mathrm{d}y = -\int_\Omega (v - w) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} w_x (v - w)_x \mathrm{d}x\mathrm{d}y = \\
&\quad -\int_R (v - w) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} (v - w) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} w_x (v - w)_x \mathrm{d}x\mathrm{d}y = \\
&\quad -\int_R (v - w) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} (v - w^\circ) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} w_x (v - w)_x \mathrm{d}x\mathrm{d}y \geq \\
&\quad -\int_R (v - w) \mathrm{d}x\mathrm{d}y + \int_{R\setminus\Omega} w_x (v - w)_x \mathrm{d}x\mathrm{d}y = \\
&\quad -\int_R (v - w) \mathrm{d}x\mathrm{d}y - \int_{R\setminus\Omega} \Delta w (v - w) \mathrm{d}x\mathrm{d}y + \int_{\partial(R\setminus\Omega)} \frac{\partial w}{\partial n} (v - w) d\Gamma \geq \\
&\quad -\int_R (v - w) \mathrm{d}x\mathrm{d}y + \int_{\partial(R\setminus\Omega)} \frac{\partial w}{\partial n} (v - w) d\Gamma.
\end{aligned}$$

Using relation (4.3) and the fact that $\varphi(x)$ is decreasing, we have on $R \cap \Gamma_\lambda$:

$$\frac{\partial w}{\partial n}(v-w) = \left(\frac{\partial w}{\partial x}\cos\alpha(e_1,n) + \frac{\partial w}{\partial y}\cos\alpha(e_2,n)\right)(v-w) =$$

$$-w_x^\circ \cos\alpha(e_1,n)(v-w) = \left(\frac{h_1^2-h_2^2}{2l} - Q(x)\right)\cos\alpha(e_1,n)(v-w) \geq 0,$$

where $e_1 = (1,0)$, $e_2 = (0,1)$ and $\alpha(\ ,\ )$ denotes the angle between the corresponding vectors. Observing that $v-w=0$ on ∂R and using assumption (3.1), we obtain:

$$\int_R (w_x(v-w)_x + w_y(v-w)_y)\mathrm{d}x\mathrm{d}y \geq -\int_R (v-w)\mathrm{d}x\mathrm{d}y.$$

Thus, we have the following:

Theorem 4.1. Let $\{\varphi, u\}$ be a solution of Problem 3.1, $\varphi(x)$ is smooth, $u \in H^1(\Omega) \cap C^\circ(\overline{\Omega})$, w is given by formula (4.1), $w_\circ(x)$ is defined by conditions (4.2), $w^\circ(x,y) \equiv w_\circ(x)$ for $(x,y) \in R$ and

$$w(x,y) = \begin{cases} w(x,y), & (x,y) \in \Omega, \\ w^\circ(x,y), & (x,y) \in R \setminus \Omega. \end{cases}$$

Then w satisfies:

$$w \in K_\circ, \quad \int_R (w_x(v-w)_x + w_y(v-w)_y)\mathrm{d}x\mathrm{d}y \geq -\int_R (v-w)\mathrm{d}x\mathrm{d}y, \quad \forall v \in K_\circ, \qquad (4.6)$$

where $K_\circ$ is defined by (4.5).

By the definition of function w°, the subset $K_\circ$ depends implicitly on the flow through the contact part of Γ_λ. This part is unknown *a priori* and is defined by the function w. Hence, (4.6), (4.5) is a quasivariational inequality, whereas for Problem 2.1 we have a variational one. We show below that, if the solution w of quasivariational inequality (5.6), (4.5) exists, then the function $u = y - w_y$, together with the curve $\varphi(x)$ that separates two regions of R, where $w = w^\circ$ and $w > w^\circ$, satisfy Problem 3.1.

5. Equivalence of the quasivariational inequality to Problem 3.1

We follow here the scheme proposed in [16, Chapter VII, Sections 3–4] for the classical case and present the proofs of the Lemmas that were subjected to essential modification in our case.

Let $w \in W^{2,p}(R) \cap C^1(\overline{R})$ with $1 \leq p < \infty$ be a solution of inequality (4.6), (4.5), define $\Omega = \{(x,y) \in R \mid w(x,y) > w^\circ(x,y)\}$, $u := y - w_y$ in Ω and assume that

$$\frac{h_\circ^2 - h_2^2}{2L} \leq -\int_0^{h_\circ} u_x(0,t)\mathrm{d}t \leq \frac{h_1^2-h_2^2}{2L}. \qquad (5.1)$$

Moreover, we suppose that the function $\varepsilon(x)$ satisfies $\varepsilon'(x) \geq 0$ and

$$\int_0^L \varepsilon(x)\mathrm{d}x \leq \frac{h_\circ^2 - h_2^2}{2L}. \qquad (5.2)$$

The assumptions (5.1) and (5.2) mean that the vertical discharge corresponding to the velocity potential u in Ω satisfies (3.1).

We use here the Hopf maximum principle for the Laplace operator, see [7, Chapter 17], the weak maximum principle and the results of the regularity theory, see [16, Chapter IV]. Let $\Gamma_4 = S_\circ \setminus \Gamma_\lambda$ and Γ_3 be defined as in Figure 3.

Lemma 5.1. The function w satisfies $w_x \leq w_x^\circ$ and $w_y \leq 0$ in R.

Proof. Since $w \in C^1(\overline{R})$, $0 < \lambda < 1$, the functions w_x and w_y are continuous in $\overline{R}$; moreover, the set Ω is open. As $\Delta(w_x - w_x^\circ) = \varepsilon'(x) \geq 0$ and $\Delta w_y = 0$ in Ω, relying on the weak maximum principle, we have that $w_x - w_x^\circ \leq \sup_{\partial\Omega}(w_x - w_x^\circ)$ and $w_y \leq \sup_{\partial\Omega} w_y$.

Using the continuity of w_y in R and the fact that $w = w^\circ$ in $R \setminus \Omega$, we obtain $w_x = w_x^\circ$ and $w_y = 0$ on $\partial\Omega \cup R$.

Since $w = w^\circ$ on $\Gamma_3 \cup \Gamma_\sigma \cup \Gamma_4 \cup S_\circ$, we have $w_x = w_x^\circ$ on $\Gamma_4 \cup S_\circ$ and $w_y = w_y^\circ = 0$ on $\Gamma_3 \cup \Gamma_\sigma$. Furthermore, as $w \geq w^\circ$ in a neighborhood $\Sigma \subset R$ of any point of $\Gamma_3 \cup \Gamma_\sigma \cup \Gamma_4 \cup S_\circ$, it follows from the definition of the partial derivative that $w_x \leq w_x^\circ$ on $\Gamma_3 \cup \Gamma_\sigma$ and $w_y \leq w_y^\circ = 0$ on $\Gamma_4 \cup S_\circ$. Moreover, $w_y = G_y \leq 0$ on $\Gamma_1 \cup \Gamma_2$. Due to definition of u, relation (4.3) and assumptions (5.1)–(5.2), we have $w_x = G_x = -\dfrac{h_1^2 - h_2^2}{2L} \leq w_x^\circ$ on $\Gamma_\circ$. Since $w_x - w_x^\circ \leq 0$ on $\Gamma_\circ$, $w(x,0) - w^\circ(x,0) \geq w(L,0) - w^\circ(L,0) = \dfrac{h_2^2}{2} > 0$ on $\Gamma_\circ$. Moreover, $\Delta w = 1$ in Ω, w is continuous and $G \in C^2(\Gamma_\circ)$. Thus, the regularity theory permits us to conclude that there exists an open neighborhood $\Sigma \subset \Omega$ with $\Gamma_\circ \subset \partial\Sigma$ such that $w \in C^2(\Sigma \cup \Gamma_\circ)$. Then $w_{xx} = G_{xx} = 0$ and $w_{yy} = 1 - w_{xx} = 1 - G_{xx} = 1$ on $\Gamma_\circ$. Hence on $\Gamma_\circ$ we have $\partial w_y/\partial n \equiv -w_{yy} = -1$. This means, due to the Hopf maximum principle, that the maximum of w_y could not be achieved on $\Gamma_\circ$. On $\partial\Omega \setminus \Gamma_\circ$, as was showed above, $w_y \leq 0$.

In a similar way, we can prove that $w_{xx} = 0$ on $\Gamma_1 \cup \Gamma_2$. Since $\Delta w_x = 0$ in Ω, $\partial w_x/\partial n \equiv -w_{xx} = 0$ on Γ_1 and $\partial w_x/\partial n \equiv w_{xx} = 0$ on Γ_2, due to the Hopf maximum principle, we conclude that the maximum of w_x could not be achieved on $\Gamma_1 \cup \Gamma_2$, whereas on the rest of the boundary we have $w_x \leq w_x^\circ$.

Hence, $w_x \leq w_x^\circ$ and $w_y \leq 0$ on Ω and, by the definition of w, everywhere over R. □

Lemma 5.2. Let be $P_\circ = (x_\circ, y_\circ) \in R$, $\Lambda^+(P_\circ) = \{(x,y) \in \overline{R} \mid x > x_\circ, y > y_\circ\}$ and $\Lambda^-(P_\circ) = \{(x,y) \in R \mid x < x_\circ, y < y_\circ\}$. If $P_\circ \in R \setminus \Omega$, then $\Lambda^+(P_\circ) \subset \overline{R} \setminus \overline{\Omega}$ and if $P_\circ \in R \cap \partial\Omega$, then $\Lambda^-(P_\circ) \subset \Omega$.

The proof is similar to [16, p. 238], taking for w the function $w - w^\circ$.

Let $\varphi(x)$ be defined as

$$\begin{aligned} &\varphi(x) = \inf\{y \mid (x,y) \in R \setminus \Omega\} \text{ for } l^\circ < x < L,\\ &\varphi(l_\circ) = \lim_{x \to (l^\circ)^+} \varphi(x), \quad \varphi(l) = \lim_{x \to L^-} \varphi(x). \end{aligned} \tag{5.3}$$

Due to Lemma 5.2, the function $\varphi(x)$ is nonincreasing. In particular, $\varphi(x)$ is nonincreasing in a neighborhood of $x = l^\circ$ and $x = L$; thus the limits above exist.

Function $u := y - w_y$ satisfies the equation $\Delta u = 0$ in Ω and the boundary conditions of Problem 3.1 on Γ_1, Γ_2 and Γ_σ, see [16, pp. 237–239]. As was shown in the proof of Lemma

5.1, $w_{yy} = 1$ on $\Gamma_\circ$; thus $q \equiv -u_y = 0$ on $\Gamma_\circ$. For the part of Γ_λ that does not contact $S_\circ$, we have $u = y$ and $q = 0$, similar to [16, p. 241]. As $w = w^\circ$ on $S_\circ$, $w_{xx} = w^\circ_{xx} = -\varepsilon(x)$ on $\Gamma_\lambda \cap S_\circ$. Since $\Delta w = 1$ in Ω, we have $w_{yy} = 1 - w_{xx} = 1 + \varepsilon(x)$ and $q \equiv u_y = -\varepsilon(x)$ on $\Gamma_\lambda \cap S_\circ$.

Lemma 5.3. The set $\partial\Omega \cap R$ does not contain segments parallel to the x or y axes. Hence φ is continuous and strictly decreasing.

The proof of this Lemma is based on contradiction and uses the Hopf maximum principle. For more details see [16, p. 240].

All these considerations lead us to the following result:

Theorem 5.1. Let $w \in W^{2,p}(R) \cap C^1(\overline{R})$, with $1 \leq p < \infty$, be a solution of quasivariational inequality (4.5). Let Ω be the following set $\{(x, y) \in R \mid w(x, y) > w^\circ(x, y)\}$ and $u := y - w_y$ in Ω satisfies (5.1). Assume that $\varepsilon(x)$ satisfies (5.2) and $\varepsilon'(x) \geq 0$, and define $\varphi(x)$ by formula (5.3). Then the pair $\{u, \varphi\}$ is the solution of Problem 3.1.

By Lemma 5.1, the assumption $\varepsilon'(x) \geq 0$ guarantees that $w_x \leq w^\circ_x$ in Ω, which due to (4.1), means that the horizontal flux is always non-negative. Thus, for the solution of Problem 3.1 obtained from (4.6)–(4.5), the fluid always flows from the left to the right of the aquifer, but not in the opposite direction.

6. Existence and uniqueness results

Replacing the value of l° in (4.6)–(4.5) by an arbitrary and fixed $l \in [0, L]$ and denoting in this case the function defined by (4.2) and (4.4) by $w^\circ_l(x, y)$, we obtain a family of variational inequalities depending on the parameter l:

$$w \in K_l, \int_R (w_x(v - w)_x + w_y(v - w)_y)\mathrm{d}x\mathrm{d}y \geq -\int_R (v - w)\mathrm{d}x\mathrm{d}y, \quad \forall v \in K_l, \tag{6.1}$$

where

$$K_l = \{v \in H^1(R) \mid v \geq w^\circ_l \text{ on } R \text{ and } v = G \text{ in } \partial R\} \tag{6.2}$$

and G is defined as in Section 4 with $w^\circ(x, y)$ substituted for $w^\circ_l(x, y)$.

The variational inequality (6.1)–(6.2) admits a unique solution of the class $W^{2,p}(R) \cap C^1(\overline{R})$, with $1 \leq p < \infty$, that we denote by w_l. Let us define

$$X_l = \{x \in [0, L] \mid w_l(x, y) > w^\circ_l(x, y),\ \forall y \in (0, h_\circ)\}$$

and

$$l^* = \sup X_l. \tag{6.3}$$

When $l \equiv l^*$, the function w_l is a solution of the quasivariational inequality (4.6)–(4.5). We prove that a unique value of l exists, that implies existence and uniqueness of the solution of (4.6)–(4.5).

Let Ω_l be the following set $\{(x, y) \in R \mid w_l(x, y) > w^\circ_l(x, y)\}$. Consider the mappings Λ_1 and $\Lambda_2 : [0, L] \longrightarrow C^1(\overline{R})$ defined by

$$\Lambda_1(l) = w_l(x, y) \quad \text{and} \quad \Lambda_2(l) = w_l(x, y) - w^\circ_l(x, y).$$

Lemma 6.1. Let l_1, l_2 be in the interval $[0, L]$ with $l_1 \le l_2$. Then, $w^\circ_{l_1} \le w^\circ_{l_2}$ in $\overline{R}$.

Proof. We note that the functions $w^\circ_{l_1}$ and $w^\circ_{l_2}$ are constant with respect to y. Let us define $W^\circ(x) = w^\circ_{l_1}(x, y) - w^\circ_{l_2}(x, y)$. From (4.2) and (4.4) we have that function W° is: (i) of the class $C^1([0, L])$; (ii) linear in $[0, l_1]$; (iii) convex in $(l_1, l_2]$; (iv) linear in $[l_2, L]$ and (v) $W^\circ(0) = W^\circ(L) = 0$.

From (ii) and (v) it follows that $W^\circ > 0$ or $W^\circ \le 0$ in $(0, l_1]$. Let us assume that $W^\circ > 0$. Then, due to (ii), we have that $W^\circ_x = C_\circ > 0$ in $(0, l_1)$. Using (i) and (iii), we obtain that $W^\circ_x \ge C_\circ > 0$ and $W^\circ > 0$ in $[l_1, l_2]$. Then, it follows from (i) and (iv) that $W^\circ_x = W^\circ_x(l_2) \ge C_\circ > 0$ and $W^\circ > 0$ in $[l_2, L]$, which are in contradiction with (v). Thus, $W^\circ \le 0$ in $(0, l_1]$.

Let us assume now that $W^\circ(l_2) > 0$. Taking into account (i) and (iii), we obtain $W^\circ_x(l_2) \ge 0$. Then, from (i) and (iv) follows that $W^\circ_x \ge 0$ and $W^\circ > 0$ in $[l_2, l]$, which are in contradiction with (v). Then, $W^\circ(l_2) \le 0$ and from (iii) we have that $W^\circ \le 0$ in $[l_1, l_2]$. Finally, from (iv) and (v), we conclude that $W^\circ \le 0$ in $[l_2, L]$. This proves the Lemma. □

Lemma 6.2. The mapping Λ_1 is nondecreasing.

Proof. Let l_1, l_2 be from the interval $[0, L]$ and $l_1 \le l_2$. We define $W = w_{l_1} - w_{l_2}$ in R and consider the subset $D = \{(x, y) \in R \mid W(x, y) > 0\}$. We assume that $D \ne \emptyset$ and take an arbitrary point $P \in D$. Since $w_{l_2} \ge w^\circ_{l_2}$ in R, due to Lemma 6.1 we have

$$w_{l_1}(P) > w_{l_2}(P) \ge w^\circ_{l_2}(P) \ge w^\circ_{l_1}(P).$$

Then, $w_{l_1}(P) > w^\circ_{l_1}(P)$ and $P \in \Omega_{l_1}$. Thus we conclude that $D \subset \Omega_{l_1}$. It follows from (4.1) that

$$\Delta w = \Delta(w_{l_1} - w_{l_2}) = 1 - \Delta w_{l_2} \ge 0 \quad \text{in } D.$$

Thus, due to the Hopf maximum principle, function W has a positive maximum on the boundary ∂D of D. On the other hand, $W = 0$ on $R \cap \partial D$ by the definition of D, and $W = 0$ on $\Gamma_\circ \cup \Gamma_1 \cup \Gamma_2$ due to the values of w_{l_1} and w_{l_2} at ∂R. Finally, it follows from Lemma 6.1 that $W = w^\circ_{l_1} - w^\circ_{l_2} \le 0$ on the rest of ∂R. This is a contradiction. Thus $D = \emptyset$, which completes the proof of the Lemma. □

Using the same idea, we can prove the following result also:

Lemma 6.3. The mapping Λ_2 is nonincreasing.
Let be $l_1, l_2 \in [0, L]$ and $l_1 < l_2$. Then, because of Lemma 6.2 and Lemma 6.3 we have

$$0 \le \left(w_{l_1} - w^\circ_{l_1}\right) - \left(w_{l_2} - w^\circ_{l_2}\right) \le \left(w_{l_2} - w^\circ_{l_1}\right) - \left(w_{l_2} - w^\circ_{l_2}\right) = w^\circ_{l_2} - w^\circ_{l_1} \text{ in } \overline{R}.$$

Moreover, it can be observed from the proof of Lemma 6.1 that $w^\circ_{l_2} - w^\circ_{l_1} \to 0$ in $\overline{R}$ as $l_1 \to l_2$. Thus we have continuity of Λ_2.

Lemma 6.4. Let l_1, l_2 be from the interval $[0, L]$ and $l_1 \le l_2$. Then $l_1^* \ge l_2^*$.

Proof. Let us assume that $x_\circ \in X_{l_2}$; then $w_{l_2}(x_\circ, y) - w^\circ_{l_2}(x_\circ, y) > 0, \forall y \in (0, h_\circ)$. Due to Lemma 6.3 we have $w_{l_1}(x_\circ, y) - w^\circ_{l_1}(x_\circ, y) \ge w_{l_2}(x_\circ, y) - w^\circ_{l_2}(x_\circ, y) > 0, \quad \forall y \in (0, h_\circ)$;

hence $x_\circ \in X_{l_1}$ and $X_{l_2} \subset X_{l_1}$. Thus, $l_1^* \geq l_2^*$. □

Theorem 6.1. There exists a unique solution of the class $W^{2,p}(R) \cap C^1(\bar{R})$, $1 \leq p < \infty$ of quasivariational inequality (4.6)–(4.5).

Proof. Let us take $l_\circ = 0$ and consider the function $w_{l_\circ}$ as the solution of (6.1)–(6.2). By definition (6.3) we have that $l_\circ^* \geq 0$. If $l_\circ^* = 0$, the function $w_{l_\circ}$ is a solution of the quasivariational inequality (4.6)-(4.5). Let us assume that $l_\circ^* > 0$. Then due to Lemma 6.4 and continuity of Λ_2 we can construct an increasing sequence $\{l_n\}$, $l_n \in [0, L]$, $n = 0, 1, 2, \ldots$ such that (i) $l_{n+1}^* \leq l_n^*,\ n = 0, 1, 2, \ldots$; (ii) $l_n \leq l_n^*,\ n = 0, 1, 2, \ldots$ and (iii) $l_n^* - l_n \to 0,\ n \to \infty$.

As the sequences $\{l_n\}$ and $\{l_n^*\}$ are monotonic on the compact set $[0, L]$, there exist l and $l' \in [0, L]$ such that

$$\lim_{n\to\infty} l_n = l \quad \text{and} \quad \lim_{n\to\infty} l_n^* = l'. \tag{6.4}$$

Since $|l - l'| \leq |l - l_n| + |l_n - l_n^*| + |l_n^* - l'|$, taking $n \to \infty$ and using (iii) and (6.4), we conclude that $l = l'$.

Let w_l be a solution of (6.1)–(6.2) corresponding to l (limit value of $\{l_n\}$) and l^* be defined by (6.3). Since Λ_2 is continuous $l_n \leq l^*$ and, due to Lemma 6.4, $l^* \leq l_n^*,\quad n = 0, 1, 2, \ldots$. As $l = l'$, we conclude that $l^* = l$. Thus, $w_l \in W^{2,p}(R) \cap C^1(\bar{R})$, $1 \leq p < \infty$ is a solution of (4.6)–(4.5).

Let us assume that there exist two different solutions w_1 and w_2 of (4.6)–(4.5). Then w_1 and w_2 satisfy (6.1)–(6.2) with $l = l_1$ and $l = l_2$, respectively. Moreover $l_1 = l_1^*$ and $l_2 = l_2^*$. Since for a fixed value of l, the solution of the variational inequality (6.1)–(6.2) is unique, we conclude that $l_1 \neq l_2$. Let us assume that $l_1 < l_2$. From Lemma 6.4 we obtain that $l_1^* \geq l_2^*$. Since $l_1 = l_1^*$ and $l_2 = l_2^*$, we have a contradiction. □

Using Theorems 5.1 and 6.1, we conclude that, if the solution of the quasivariationl inequality (4.6)–(4.5) satisfies conditions (5.1) and (5.2), this is a unique solution of Problem 3.1. An additional result, involving Theorems 4.1 and 5.1 is: if the solution of Problem 3.1 satisfies conditions (5.1) and (5.2), then this is a unique solution. A trivial situation, when conditions (5.1) and (5.2) hold, is the case of $\varepsilon(x) \equiv 0$. Then relations (4.6)–(4.5) become a variational inequality and Theorem 5.1 gives us the existence and uniqueness of the solution of Problem 3.1.

7. Numerical approach and test examples

An equivalent formulation of Problem 3.1 can be given in terms of a shape optimization problem for a system governed by the Laplace equation. Let Φ be the set of all feasible shapes of the water table, formed by smooth curves. The optimization problem consists in finding $\psi \in \Phi$ and u such that:

$$
\begin{cases}
\min\limits_{\psi\in\Phi} \displaystyle\int\limits_{\Gamma_\lambda \setminus S_\circ} (q)^2 \mathrm{d}\Gamma, \\
\text{where } q = \partial u/\partial n \text{ and } u(x, y) \text{ is a solution of problem:} \\
\qquad \begin{cases}
\Delta u = 0 & \text{in} \quad \Omega, \\
u = h_1 & \text{on} \quad \Gamma_1, \\
u = h_2 & \text{on} \quad \Gamma_2, \\
u = y & \text{on} \quad \Gamma_\sigma \cup (\Gamma_\lambda \setminus S_\circ), \\
q = 0 & \text{on} \quad \Gamma_\circ, \\
q = -\varepsilon(x) & \text{on} \quad \Gamma_\lambda \cap S_\circ.
\end{cases}
\end{cases}
\tag{7.1}
$$

The objective functional contains the square of the flux along the free part of the water table. The choice of the optimal water-table location forces the objective to be zero and vice versa.

This shape-optimization formulation of Problem 3.1 interprets the water table Γ_λ as an optimal boundary. The concept of the optimal boundary includes the values at the contour of the domain only. Thus, it is not necessary to solve the problem in the whole domain Ω to find the optimal boundary. On the other hand, finding Γ_λ, we can obtain $u(x, y)$ in Ω by solving the boundary-value problem (7.1). For this reason, we are looking for the location of the water table only.

In the two-dimensional case for the problem governed by the Laplace equation the values of flux and potential satisfy on the boundary $\Gamma \equiv \partial\Omega$ the integral equation, [17, Chapter 2]:

$$\frac{1}{2}u(\xi) + \int_\Gamma q^*(\xi, \chi)u(\chi)\mathrm{d}\Gamma = \int_\Gamma u^*(\xi, \chi)q(\chi)\mathrm{d}\Gamma,$$

where $\chi \equiv (x, y) \in \Gamma$, $u^*(\xi, \chi)$ is the fundamental solution of the Laplace equation, $q^*(\xi, \chi)$ its normal derivative, and $\xi \in \Gamma$ is the collocation point.

In this way, to define the location of the water table, we have the problem:

$$
\begin{cases}
\min\limits_{\psi\in\Phi} F(u, q), \\
\text{where } q \text{ and } u \text{ satisfy } \Gamma \text{ the integral equation:} \\
\frac{1}{2}u(\xi) + \displaystyle\int_\Gamma q^*(\xi, \chi)u(\chi)\mathrm{d}\Gamma = \int_\Gamma u^*(\xi, \chi)q(\chi)\mathrm{d}\Gamma,
\end{cases}
\tag{7.2}
$$

where $F(u, q) = \displaystyle\int\limits_{\Gamma_\lambda \setminus S_\circ} (q)^2 \mathrm{d}\Gamma$ and the boundary values are defined as in problem (7.1).

Formulation (7.2) furnishes an opportunity to use a boundary-element discretization. For the discrete analog of problem (7.2) we consider as independent variables the flux at the boundary elements of Γ_1, the potential at the boundary elements of $\Gamma_\circ$, the flux at the boundary elements of Γ_2, Γ_σ and $\Gamma_\lambda \setminus S_\circ$, the the potential at the boundary elements of $\Gamma_\lambda \cup S_\circ$, y-coordinates of the seepage surface nodes, y- and x-coordinates of the water-table nodes. Then, we get a nonlinear mathematical programming problem. To solve it we use Herskovits's interior-point algorithm, [14]. We find the y-coordinates of free part of the water-table and

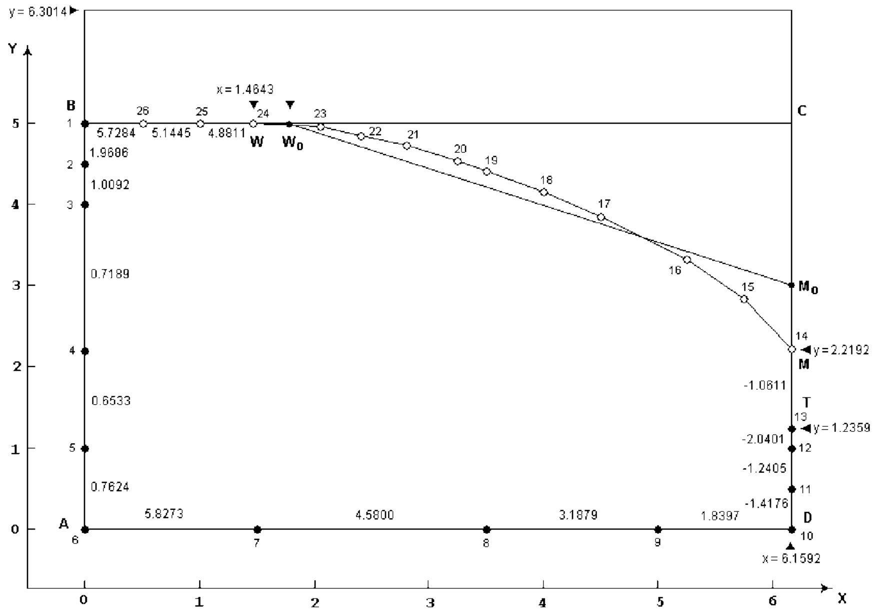

Figure 4. B.E.M. dicretization.

seepage-surface nodes, as well as the x-coordinates of the contact part of the water table and values of potential and flux at the corresponding segments of the boundary.

As an example we present here the numerical result from [15]. For the test problem we choose: $h_1 = 6{\cdot}3014$, $h_2 = 1{\cdot}2359$, $L = 6{\cdot}1592$ and $d = 1{\cdot}3014$ ($h_\circ = 5{\cdot}0$). These data are taken in order to compare the solution of the forest-impact problem with the seepage one considered in [18]. The suction flux is taken as $\varepsilon = 1$.

The discretization includes 26 boundary elements; see Figure 4. We are looking for the y-coordinates of ten nodes (14–23) at the free part of the water table $W - M$ and the x-coordinates of three nodes (24–26) at the contact part of the water table $B - W$. The position of node 24 defines the location of the contact point of the water table (point W). The coordinates of the other nodes are fixed. The water-table initial position, used at the first iteration of the algorithm, is given by the line $B - W_\circ - M_\circ$ in Figure 4.

The mathematical program has 39 variables, 26 nonlinear equality constraints, 12 'box' constraints and 2 linear inequality constrains. We adopt the algorithm stopping criterion with precision 10E−6; see [14] for details. For the different initial data, convergence of the algorithm was obtained in no more than 20 iterations.

The coordinates of the water-table nodes and the values of flux and potential calculated at the corresponding boundary elements of the water table are given in Table 1. In this table the first column indicates the node number, second and third present the x- and y-coordinates of the water-table nodes obtained numerically, the fourth column shows the value of the flux (potential) calculated at the corresponding boundary elements. Table 2 shows the iterations history: the first column gives the number of iterations, the second shows the objective function value, the third presents the maximal error in the equality constrains that corresponds to the residual error of the discrete boundary-integral equation. Figure 4 also shows the location of the water table (continuous line $B - W - M$) and positions of the corresponding nodes

Table 1. Water table coordinates and boundary values.

node	x *fixed value	y *fixed value	q ⋆value of u
14	6·1592*	2·2192	$-1{\cdot}17240\times10^{-7}$
15	5·7500*	2·8368	$-2{\cdot}82183\times10^{-8}$
16	5·2500*	3·3175	$2{\cdot}23091\times10^{-7}$
17	4·5000*	3·8532	$5{\cdot}45910\times10^{-7}$
18	4·0000*	4·1583	$-1{\cdot}23321\times10^{-7}$
19	3·5000*	4·4158	$-2{\cdot}90582\times10^{-7}$
20	3·2500*	4·5345	$2{\cdot}36618\times10^{-7}$
21	2·8000*	4·7249	$-2{\cdot}24024\times10^{-7}$
22	2·4000*	4·8467	$-2{\cdot}45834\times10^{-7}$
23	2·0500*	4·9553	$3{\cdot}35504\times10^{-7}$
24	1·4643	5·0000*	4·88110⋆
25	0·9270	5·0000*	5·14453⋆
26	0·4977	5·0000*	5·72835⋆

Table 2. Iterative history

Iter.	$\int_{\Gamma_\lambda \setminus S_\circ} (q)^2 d\Gamma$	Equality cnstr.
1	1·02936	3·12027
2	1·14977	2·35687
3	$8{\cdot}44161\times10^{-1}$	1·71197
4	$6{\cdot}31515\times10^{-1}$	$7{\cdot}37943\times10^{-1}$
5	$5{\cdot}66052\times10^{-1}$	$2{\cdot}76270\times10^{-1}$
6	$3{\cdot}66079\times10^{-1}$	$1{\cdot}96013\times10^{-1}$
7	$2{\cdot}59572\times10^{-1}$	$8{\cdot}91051\times10^{-2}$
8	$1{\cdot}18700\times10^{-1}$	$4{\cdot}82199\times10^{-2}$
9	$7{\cdot}08363\times10^{-3}$	$2{\cdot}72489\times10^{-2}$
10	$3{\cdot}35875\times10^{-3}$	$7{\cdot}91583\times10^{-3}$
11	$3{\cdot}72563\times10^{-3}$	$2{\cdot}48889\times10^{-3}$
12	$5{\cdot}91536\times10^{-4}$	$1{\cdot}15717\times10^{-3}$
13	$5{\cdot}95424\times10^{-5}$	$2{\cdot}52348\times10^{-4}$
14	$3{\cdot}31008\times10^{-6}$	$7{\cdot}45031\times10^{-5}$
15	$2{\cdot}94575\times10^{-7}$	$2{\cdot}25893\times10^{-5}$
16	$5{\cdot}04176\times10^{-9}$	$6{\cdot}79022\times10^{-6}$
17	$2{\cdot}09812\times10^{-10}$	$8{\cdot}67709\times10^{-7}$
18	$2{\cdot}63430\times10^{-11}$	$8{\cdot}59921\times10^{-9}$
19	$6{\cdot}91086\times10^{-13}$	$4{\cdot}60299\times10^{-10}$

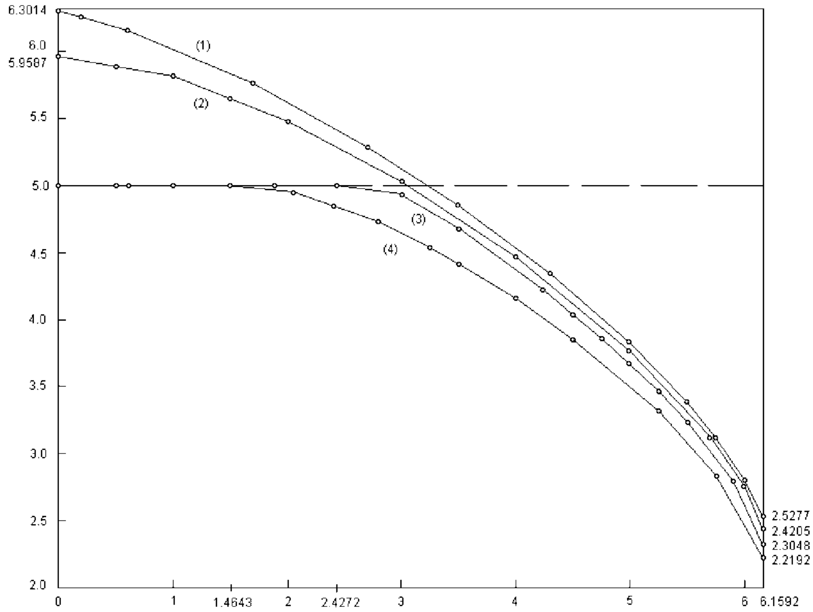

Figure 5. Water-table location. Numerical results.

(14–26) calculated numerically as well as boundary data, *i.e.*, flux at the segments $A - B$, $D - T$ and $T - M$ and potential for the segments $A - D$ and $B - W$.

We compare the location of the water table in the forest-impact problem with the solution of other unconfined problems, with the same geometrical and piezometric parameters. The first one is the classical seepage problem; Figure 2. We consider also the situation when only the vertical impermeable wall Γ_w is present. Finally, we solve the forest-impact problem assuming that the bottom $S_\circ$ is impermeable, *i.e.*, the suction rate $\varepsilon = 0$. The results are presented in Figure 5. Here line (1) defines the location of the water table for the classical seepage problem, line (2) gives the location of the water table for the unconfined problem with vertical impermeable wall, line (3) is the water table in the case of an impermeable bottom $S_\circ$, line (4) is the solution of the forest-impact problem with constant suction rate $\varepsilon = 1$. We can observe in these examples that forest suction provides sufficient lowering of the groundwater table.

8. Conclusions

Considering the forest-impact phenomenon, we introduce restrictions on the water table that combine free-boundary and contact-boundary conditions. Thus the problem we have is a 'free-contact' boundary problem. The variational formulation that we obtain for this problem is a quasivariational inequality. Because of the quasivariational character of the inequality, the analysis of the existence and uniqueness of the solution of the forest-impact problem and its numerical simulation require techniques that are different from the approaches used for classical seepage problems. The numerical results show that, even for our model of forest impact on aquifers, wich takes into account some principal characteristics of this phenomenon only, the water-table lowering owing to forest suction is significant enough to be considered as an effective means for the control of groundwater.

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